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ON CERTAIN THEOREMS OF LIAPUNOV'S SECOND METHOD

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We give sufficient conditions for asymptotic stability relative to a part of the variables. We investigate the question of the invertibility of certain proved and well-known theorems of Liapunov's second method. With the aid of the Liapunov function method we give the necessary and sufficient conditions for the boundedness of solutions relative to a part of the variables.

1. Let us consider a system of differential equations of perturbed motion

$$\mathbf{x}^{\cdot} = \mathbf{X}(t, \mathbf{x}) \qquad (\mathbf{X}(t, 0) \equiv 0) \tag{1.1}$$

$$\mathbf{x} = (x_1, \ldots, x_n), \qquad \mathbf{X} = (X_1, \ldots, X_n)$$

We shall study the question of the stability of the unperturbed motion $\mathbf{x} = 0$ relative to $x_1, \ldots, x_m$ $(0 < m \leqslant n)$. Denoting these variables by $y_i = x_i$ $(i = 1, \ldots, m)$, and the remaining by $z_j = x_{m+j}$ $(j = 1, \ldots, n - m = p)$, i.e. $\mathbf{x} = (y_1, \ldots, y_m, z_1, \ldots, z_p)$ we introduce the notation

$$\|\mathbf{y}\| = \Big(\sum_{i=1}^{m} y_i^2\Big)^{1/2}, \qquad \|\mathbf{z}\| = \Big(\sum_{j=1}^{p} z_j^2\Big)^{1/2}$$

$$\|\mathbf{x}\| = \Big(\sum_{i=1}^{n} x_i^2\Big)^{1/2} = (\|\mathbf{y}\|^2 + \|\mathbf{z}\|^2)^{1/2}$$

We assume that:

a) in the region

$$t \geqslant 0, \qquad \|\mathbf{y}\| \leqslant H > 0, \qquad 0 \leqslant \|\mathbf{z}\| < +\infty \tag{1.2}$$

the right hand sides of system (1.1) are continuous and satisfy the conditions for the uniqueness of the solution;

b) the solutions of system (1.1) are $\mathbf{z}$-extendable; this means that any solution $\mathbf{x}(t)$ is defined for all $t \geqslant 0$ for which $\|\mathbf{y}(t)\| \leqslant H$.

By $\mathbf{x} = \mathbf{x}(t; t_0, \mathbf{x}_0)$ we denote the solution of system (1.1) defined by the initial conditions $\mathbf{x}(t_0; t_0, \mathbf{x}_0) = \mathbf{x}_0$.

Theorem 1. If there exists a function $V(t, \mathbf{x})$ satisfying the conditions:

1)
$$V(t, \mathbf{x}) \geqslant a(\|\mathbf{y}\|) \tag{1.3}$$

where $a(r)$ is a continuous monotonically-increasing function and $a(0) = 0$;

2) $V^{\cdot} \leqslant 0$ by virtue of (1.1) and for any $\eta > 0$

$$V^{\cdot}(\tau, \mathbf{x}) \leqslant -m_\eta(\tau) \tag{1.4}$$

follows from $V(\tau, \mathbf{x}) \geqslant \eta$, $\|\mathbf{y}\| \leqslant H$, where

$$\int_0^\infty m_\eta(\tau)\, d\tau = +\infty \tag{1.5}$$

then the motion $\mathbf{x} = 0$ is asymptotically $\mathbf{y}$-stable. If, further, system (1.1) and the function V are ω-periodic in t (or are independent of t), then the asymptotic y-stability is uniform in $\{t_0, \mathbf{x}_0\}$.

Proof. The hypotheses of the $\mathbf{y}$-stability theorem [1] are satisfied, therefore, for any $\varepsilon \in (0, H)$, $t_0 \geqslant 0$ we can find $\delta(\varepsilon, t_0) > 0$ such that from $\|\mathbf{x}_0\| < \delta$ it follows that $\|\mathbf{y}(t, t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geqslant t_0$. Let us show that when $\|\mathbf{x}_0\| < \delta$,

$$\lim_{t\to\infty} V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) = 0 \tag{1.6}$$

Otherwise, because $V^{\cdot} \leqslant 0$ we would have $V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \geqslant \eta > 0$ and from

$$V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) = V(t_0, \mathbf{x}_0) + \int_{t_0}^{t} V^{\cdot}(\tau, \mathbf{x}(\tau; t_0, \mathbf{x}_0))\, d\tau \tag{1.7}$$

would follow

$$0 \leqslant V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \leqslant V(t_0, \mathbf{x}_0) - \int_{t_0}^{t} m_\eta(\tau)\, d\tau$$

which is impossible for t sufficiently large because of (1.5). The asymptotic y-stability of the motion $\mathbf{x} = 0$ follows from (1.6). When system (1.1) and the function V are ω-periodic in t, the required uniformity follows from Theorem 1 of [2].

Theorem 2. If there exists a function $V(t, \mathbf{x})$ satisfying the conditions:

$$1) \quad a(\|\mathbf{y}\|) \leqslant V(t, \mathbf{x}) \leqslant b\left(\left(\sum_{i=1}^{k} x_i^2\right)^{1/2}\right), \qquad m \leqslant k \leqslant n \tag{1.8}$$

2) (1.4) and (1.5) follow from

$$\sum_{i=1}^{k} x_i^2 \geqslant \eta^2, \qquad \|\mathbf{y}\| \leqslant H$$

for any $\eta > 0$, then the motion $\mathbf{x} = 0$ is asymptotically $\mathbf{y}$-stable uniformly in $\mathbf{x}_0$ from the region (*)

$$\sum_{i=1}^{k} x_{i0}^2 < \delta^2, \quad -\infty < x_{j0} < +\infty \quad (j = k+1, \ldots, n), \quad \delta = \text{const} > 0 \tag{1.9}$$

Proof. Set $\delta = b^{-1}(a(H))$. If (1.9) is satisfied,

$$a(\|\mathbf{y}(t; t_0, \mathbf{x}_0)\|) \leqslant V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \leqslant V(t_0, \mathbf{x}_0) \leqslant b\left(\left(\sum_{i=1}^{k} x_{i0}^2\right)^{1/2}\right) < a(H)$$

whence $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| < H$ for $t \geqslant t_0$ and, consequently, the solution $\mathbf{x}(t; t_0, \mathbf{x}_0)$ is defined for all $t \in [t_0, \infty)$. For every $\varepsilon > 0$, $t_0 \geqslant 0$ there exists, by virtue of (1.5), $T(\varepsilon, t_0) > 0$ such that for $\eta = h \equiv b^{-1}(a(\varepsilon))$

$$\int_{t_0}^{t_0+T} m_h(\tau)\, d\tau = a(H) \tag{1.10}$$

If we assume that $V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \geqslant a(\varepsilon)$ for all $t \in (t_0, t_0 + T)$, then by virtue of (1.4) and (1.10), from (1.7) would follow

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*) This means that for a certain $\delta > 0$ there exists, for any $\varepsilon > 0$, $t_0 \geqslant 0$, a $T(\varepsilon, t_0) > 0$ such that $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geqslant t_0 + T$, if $\mathbf{x}_0$ lies in region (1.9).

$$0 < a(\varepsilon) \leqslant V(t_0 + T,\ \mathbf{x}(t_0 + T;\ t_0,\ \mathbf{x}_0)) \leqslant V(t_0,\ \mathbf{x}_0) - \int_{t_0}^{t_0+T} m_h(\tau)\, d\tau \leqslant a(H) - $$
$$- \int_{t_0}^{t_0+T} m_h(\tau)\, d\tau = 0$$

which is impossible. Consequently, for some $t_* \in (t_0,\ t_0 + T)$ we have $V(t_*,\ \mathbf{x}(t_*;\ t_0,\ \mathbf{x}_0)) < a(\varepsilon)$. Since $V^\cdot \leqslant 0$, then for $t \geqslant t_*$

$$a(\|\mathbf{y}(t;\ t_0,\ \mathbf{x}_0)\|) \leqslant V(t,\ \mathbf{x}(t;\ t_0,\ \mathbf{x}_0)) \leqslant V(t_*,\ \mathbf{x}(t_*;\ t_0,\ \mathbf{x}_0)) < a(\varepsilon)$$

whence $\|\mathbf{y}(t;\ t_0,\ \mathbf{x}_0)\| < \varepsilon$ for all $t \geqslant t_0 + T > t_*$. The theorem is proved.

Note. The identities

$$X_i(t,\ 0, \ldots, 0,\ x_{k+1}, \ldots, x_n) \equiv 0 \qquad (i = 1, \ldots, m)$$

are necessary for the fulfillment of the hypotheses of Theorem 2 and are proved analogously to [4].

Theorems 1 and 2 generalize the results of [3]. When $m < n$ these theorems cannot be inverted even for autonomous systems which are asymptotically $\mathbf{y}$-stable uniformly in $\{t_0,\ \mathbf{x}_0\}$ as shown by the following example.

Consider a system [4]

$$x^\cdot = -x\varphi(y), \qquad y^\cdot = 0 \tag{1.11}$$

in which $\varphi(y)$ is a smooth function, where $\varphi(y) > 0$ for $|y| < 1$, $\varphi(y) \equiv 0$ for $|y| \geqslant 1$. 1. The solution $x = y = 0$ of system (1.11) is asymptotically x-stable uniformly in $\{t_0,\ x_0,\ y_0\}$ [4].

Let us show that a function V satisfying the hypotheses of Theorem 1 (*) does not exist for system (1.11). We assume the contrary: suppose that $V(t,\ x,\ y) \geqslant a(|x|)$, but that $V^\cdot(\tau,\ x,\ y) \leqslant -m_\eta(\tau)$ follows from $V(\tau,\ x,\ y) \geqslant \eta > 0$, $|x| \leqslant H$ and (1.5) holds. In the region $|y| \geqslant 1$, $V^\cdot \equiv \partial V / \partial t$. Because this region is convex in t we have ([5], p.154)

$$V(t,\ x,\ y) = \int_0^t \frac{\partial}{\partial \tau} V(\tau,\ x,\ y)\, d\tau + \psi(x,\ y)$$

whence follows, for $x \neq 0$

$$0 \leqslant V(t,\ x,\ y) \leqslant -\int_0^t m_{a(|x|)}(\tau)\, d\tau + \psi(x,\ y)$$

which is impossible for t sufficiently large.

Under the assumption of continuity and boundedness of the derivatives $\partial X_i / \partial x_j$ a theorem inverse to Theorem 1 was stated in [3] for the case $m = n$. If the derivatives $\partial X_i / \partial x_j$ are continuous, but not bounded, the inverse theorem does not hold, as shown by the example of the scalar equation [8]

$$x^\cdot = -x\varphi(t,\ x) \tag{1.12}$$

in which φ is a smooth function, and $\varphi = 1$ for $|x| \leqslant e^{-t}$ and $\varphi = 0$ for $|x| \geqslant 2e^{-t}$. The solution $x = 0$ of Eq. (1.12) is asymptotically stable [6]. Let us show that a function $V(t,\ x)$ satisfying the hypotheses of Theorem 1 does not exist for this equation. We assume the contrary: suppose that $V(t,\ x) \geqslant a(|x|)$, but that $V^\cdot(\tau,\ x) \leqslant -m_\eta(\tau)$ follows

*) The proof is carried out analogously for Theorem 2.

from $V(\tau, x) \geqslant \eta > 0$, $|x| \leqslant H$, and (1.5) holds. In the region $2e^{-t} \leqslant |x| \leqslant H$ we have $V^{\cdot} \equiv \partial V / \partial t$, therefore [6],

$$a(|x|) \leqslant V(t, x) = \int_{\tau(x)}^{t} \frac{\partial V(\xi, x)}{\partial \xi} d\xi + \psi(x) \leqslant - \int_{\tau(x)}^{t} m_{a(|x|)}(\xi)\, d\xi + \psi(x)$$

$$(\tau(x) = -\ln(^1/_2 |x|))$$

which, by virtue of (1.5), is impossible for t sufficiently large.

2. Theorem 3. If in the region

$$t \geqslant 0, \qquad \|\mathbf{x}\| \leqslant H > 0 \tag{2.1}$$

the right hand sides of system (1.1) are uniformly bounded

$$\|\mathbf{X}(t, \mathbf{x})\| \leqslant N \qquad (N = \mathrm{const} > 0) \tag{2.2}$$

and if there exists a function $V(t, \mathbf{x})$ such that $V \geqslant 0$, while its derivative by virtue of system (1.1)

$$V^{\cdot}(t, \mathbf{x}) \leqslant -c(\|\mathbf{x}\|) \tag{2.3}$$

($c(r)$ is a function of the type of $a(r)$), then $V(t, \mathbf{x})$ is a positive-definite function.

Proof. From (2.2) it follows that the solution $\mathbf{x}(t; t_0, \mathbf{x}_0)$ with initial point $(t_0, \mathbf{x}_0)$ from the region

$$\|\mathbf{x}\| \leqslant H_1 \equiv {}^2/_3 H, \qquad t \geqslant 0 \tag{2.4}$$

is defined for $0 \leqslant t - t_0 \leqslant H_1 / (2N)$ and satisfies the inequality

$$\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| \leqslant H \tag{2.5}$$

Let us show that V is a function which is positive definite in region (2.4). We assume the contrary: suppose that for some ε_0, $0 < \varepsilon_0 < H_1$, for any arbitrarily small $\delta > 0$ we can find a point $(t_*, \mathbf{x}_*)$, $t_* \geqslant 0$, $\varepsilon_0 \leqslant \|\mathbf{x}_*\| \leqslant H_1$, for which $V(t_*, \mathbf{x}_*) < \delta$. We have

$$V(t_*, \mathbf{x}_*) < \frac{\varepsilon_0}{2N} c\left(\frac{\varepsilon_0}{2}\right) \quad \text{for} \quad \delta < \frac{\varepsilon_0}{2N} c\left(\frac{\varepsilon_0}{2}\right) \tag{2.6}$$

From (2.2) and inequality $\|\mathbf{x}_*\| \geqslant \varepsilon_0$ follows

$$\|\mathbf{x}(t; t_*, \mathbf{x}_*)\| \geqslant \frac{\varepsilon_0}{2} \quad \text{for} \quad 0 \leqslant t - t_* \leqslant \frac{\varepsilon_0}{2N} < \frac{H_1}{2N} \tag{2.7}$$

By virtue of (2.6) and (2.7),

$$0 \leqslant V\left(t_* + \frac{\varepsilon_0}{2N}, \mathbf{x}\left(t_* + \frac{\varepsilon_0}{2N}; t_*, \mathbf{x}_*\right)\right) \leqslant V(t_*, \mathbf{x}_*) - \frac{\varepsilon_0}{2N} c\left(\frac{\varepsilon_0}{2}\right) < 0$$

follows from (1.7) for the instant $t = t_* + \varepsilon_0 / (2N)$ (in view of (2.5) the solution is still defined at this t), which is impossible. The theorem is proved.

Note. Condition (2.2) is satisfied, for example, if system (1.1) is periodic in t (or autonomous).

Lemma. If there exists a function $V(t, \mathbf{x})$ such that $V \geqslant 0$ in region (2.1), while $V^{\cdot} \leqslant 0$, the inequality $V(t, \mathbf{x}) > 0$ is fulfilled at each point $(t, \mathbf{x})$ at which $V^{\cdot}(t, \mathbf{x}) < 0$.

Proof. If it should be that at some point $(t_*, \mathbf{x}_*)$ we have $V^{\cdot}(t_*, \mathbf{x}_*) < 0$ but $V(t_*, \mathbf{x}_*) = 0$, for a sufficiently small $\varepsilon > 0$, $V < 0$ would follow from

$$V(t_* + \varepsilon, \mathbf{x}(t_* + \varepsilon; t_*, \mathbf{x}_*)) = V(t_*, \mathbf{x}_*) + \int_{t_*}^{t_*+\varepsilon} V^{\cdot}(\tau, \mathbf{x}(\tau; t_*, \mathbf{x}_*))\, d\tau = V^{\cdot}(t_*, \mathbf{x}_*)\,\varepsilon + o(\varepsilon)$$

which is impossible.

Theorem 4. If in region (2.1):

1) $V(t, \mathbf{x}) \geqslant 0$

2) the function V is periodic in t (or is independent of time)

3) $V^{\cdot}(t, \mathbf{x}) < 0$ follows from $\| \mathbf{x} \| \neq 0$,

then V is a positive-definite function.

Proof. By virtue of the lemma, from conditions (1) and (3) it follows that $V(t, \mathbf{x}) > 0$ for $\| \mathbf{x} \| \neq 0$. Therefore, the positive definiteness of function V follows from condition (2) [7].

Note. Condition (3) is fulfilled, for example, if $V^{\cdot}$ is a negative-definite function. Theorem 4 is not true if we omit condition (2), as the following example shows: for the equation $x^{\cdot} = -xe^{t}$ the constantly-positive function $V = x^2 e^{-t}$, which is not positive definite has a negative-definite derivative.

The following is well known:

Theorem A [8 – 10]. If there exists a function $V(t, \mathbf{x})$ satisfying the conditions

$$a(\| \mathbf{y} \|) \leqslant V(t, \mathbf{x}) \leqslant b(\| \mathbf{x} \|)$$

in region (1.2) and if (2.3) holds, the motion $\mathbf{x} = 0$ is asymptotically $\mathbf{y}$-stable uniformly in $\{t_0, \mathbf{x}_0\}$.

From Theorems 3 and 4 it follows that if a function V exists satisfying the hypotheses of Theorem A and if one of the next two conditions are fulfilled: either (2.2) holds (in region (2.1)) or V is periodic in t, then the function V is necessarily positive definite and, consequently, the motion $\mathbf{x} = 0$ is asymptotically Liapunov-stable (uniformly in $\{t_0, \mathbf{x}_0\}$ [7]). Thus, a function V, which is not positive-definite in all the variables and which satisfies the hypotheses of Theorem A (for example when there is no asymptotic Liapunov-stability), can exist only when system (1.1) and function V depend "essentially" on time. For example, for the system $x^{\cdot} = -x + ye^{-t}$, $y^{\cdot} = -x - ye^{-t}$ the x-positive-definite function $V = x^2 + y^2 e^{-t}$, admitting of an infinitesimal upper bound, is not positive-definite in (x, y) but has a negative-definite derivative.

3. The Liapunov function method can be applied to investigate the boundedness of solutions [11 – 14]. Analogous results hold in the problem of $\mathbf{y}$-boundedness.

We assume that the right hand sides of system (1.1) are continuous and satisfy the conditions for the uniqueness of the solution in the region

$$0 \leqslant \| \mathbf{x} \| < +\infty, \qquad t \geqslant 0 \tag{3.1}$$

moreover, it is not necessary that $\mathbf{X}(t, 0) \equiv 0$; here $\mathbf{z}$-extendability signifies that any solution $\mathbf{x}(t; t_0, \mathbf{x}_0)$ is defined for all $t \geqslant 0$ for which $\| \mathbf{y}(t; t_0, \mathbf{x}_0) \| < +\infty$.

Definitions. The solutions of system (1.1) are said to be:

a) $\mathbf{y}$-bounded if for any $t_0 \geqslant 0$, $\mathbf{x}_0$ we can find $N(t_0, \mathbf{x}_0) > 0$ such that for $t \geqslant t_0$

$$\| \mathbf{y}(t; t_0, \mathbf{x}_0) \| \leqslant N \tag{3.2}$$

b) $\mathbf{y}$-bounded uniformly in t_0 if in (a) we can choose $N(\mathbf{x}_0) > 0$ independent of t_0 for any $\mathbf{x}_0$;

c) $\mathbf{y}$-bounded uniformly in $\mathbf{x}_0$ if for any $t_0 \geqslant 0$ and a compactum K of the space $\{x_1, \ldots, x_n\}$ we can find $N(t_0, K) > 0$ such that (3.2) follows from $\mathbf{x}_0 \in K$, $t \geqslant t_0$;

d) $\mathbf{y}$-bounded uniformly in $\{t_0, \mathbf{x}_0\}$ if in (c) we can choose $N(K) > 0$ independent of t_0 for any compactum K.

Theorem 5. In order for the solutions of system (1.1) to be:

1) y-bounded, it is necessary and sufficient that there exists a function $V(t, x)$ satisfying inequality (1.3) in region (3.1), where $a(\|y\|) \to +\infty$ as $\|y\| \to \infty$ and the function $V(t, x(t; t_0, x_0))$ does not grow for any solution $x(t; t_0, x_0)$;

2) y-bounded uniformly in t_0, it is necessary and sufficient that there exists a function V satisfying the conditions (1) and, further, the inequality

$$V(t, x) \leqslant W(x) \tag{3.3}$$

where $W(x)$ is a function (discontinuous, in general) which is finite at every point x;

3) y-bounded uniformly in x_0, it is necessary and sufficient that there exists a function V satisfying the conditions in (1) and such that for any compactum K

$$V(t, x) \leqslant \varphi_K(t) \quad \text{for} \quad x \in K, \quad t \geqslant 0 \tag{3.4}$$

4) y-bounded uniformly in $\{t_0, x_0\}$, it is necessary and sufficient that there exists a function V satisfying the conditions in (1) and the inequality

$$V(t, x) \leqslant b(\|x\|) \tag{3.5}$$

where $b(r)$ is a function increasing monotonically for $r \in [0, \infty)$ (*).

Proof. 1) Sufficiency. For $V_0 \equiv V(t_0, x_0)$ there exists $N(V_0) = N(t_0, x_0) > 0$ such that $a(\|y\|) > V_0$ follows from $\|y\| > N$. Further, we have

$$a(\|y(t; t_0, x_0)\|) \leqslant V(t, x(t; t_0, x_0)) \leqslant V_0$$

whence $\|y(t; t_0, x_0)\| \leqslant N$ for $t \geqslant t_0$.

Necessity. The function

$$V(t, x) = \sup_{\tau \geqslant 0} \|y(t+\tau; t, x)\| \tag{3.6}$$

is defined by virtue of the y-boundedness in region (3.1). Obviously, $V(t, x) \geqslant \|y\|$ If $t_1 < t_2$, then

$$V(t_1, x(t_1; t_0, x_0)) = \sup_{\tau \geqslant 0} \|y(t_1+\tau; t_0, x_0)\| \geqslant \sup_{\tau \geqslant 0} \|y(t_2+\tau; t_0, x_0)\| = V(t_2, x(t_2; t_0, x_0))$$

i.e., $V(t, x(t; t_0, x_0))$ does not grow.

2) Sufficiency. We choose $N(x_0) > 0$ such that $a(\|y\|) > W(x_0)$ follows from $\|y\| > N$. In this case (see (3.3))

$$a(\|y(t; t_0, x_0)\|) \leqslant V(t, x(t; t_0, x_0)) \leqslant V(t_0, x_0) \leqslant W(x_0)$$

whence $\|y(t; t_0, x_0)\| \leqslant N$ for $t \geqslant t_0$.

Necessity. The function V defined by formula (3.6) satisfies, in accord with Definition (b), the inequality $V(t, x) \leqslant N(x)$.

3) Sufficiency. For every t_0 and compactum K there exists $N(t_0, K) > 0$ such that $a(\|y\|) > \varphi_K(t_0)$ follows from $\|y\| > N$. For $x_0 \in K$, $t \geqslant t_0$ we have (see (3.4))

$$a(\|y(t; t_0, x_0)\|) \leqslant V(t, x(t; t_0, x_0)) \leqslant V(t_0, x_0) \leqslant \varphi_K(t_0)$$

whence $\|y(t; t_0, x_0)\| \leqslant N$.

Necessity. The function V defined by formula (3.6) satisfies for $x \in K$ the inequality

$$V(t, x) \leqslant N(t, K) \equiv \varphi_K(t)$$

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*) Results close to (3) and (4) of this theorem (on the sufficient conditions side) have been obtained by Peiffer, K. La méthode directe de Liapounoff appliquée à l'étude de la stabilite partielle (Dissertation). Universite Catholique de Louvain, Faculté des sciences, 1968.

4) Sufficiency. For each compactum K we denote

$$b_K = \sup [V(t, \mathbf{x}): t \geqslant 0, \mathbf{x} \in K] \leqslant \sup [b(\|\mathbf{x}\|): \mathbf{x} \in K] < +\infty$$

There exists $N(K) > 0$ such that $a(\|\mathbf{y}\|) > b_K$ follows from $\|\mathbf{y}\| > N$. Then

$$a(\|\mathbf{y}(t; t_0, \mathbf{x}_0)\|) \leqslant V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \leqslant V(t_0, \mathbf{x}_0) \leqslant b_K \quad \text{for} \quad t_0 \geqslant 0, \mathbf{x}_0 \in K$$

whence $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| \leqslant N$ for all $t \geqslant t_0$.

Necessity. For function (3.6), by selecting as compactum K the spheres $\|\mathbf{x}\| = r$, $r \in [0, \infty)$, we obtain, in accord with (d),

$$V(t, \mathbf{x}) \leqslant N(K) \equiv N(r) \quad \text{for } \mathbf{x} \in K$$

The function $N(r)$ may be considered to increase monotonically with $r \in [0, \infty)$; after this it remains to set $b(\|\mathbf{x}\|) = N(\|\mathbf{x}\|)$. The theorem is proved.

Note. Condition (3.4) is satisfied if $V(t, x)$ is continuous.

Example. For the mechanical system [1, 2, 9]

$$\frac{d}{dt}\frac{\partial T}{\partial q_i^{\cdot}} - \frac{\partial T}{\partial q_i} = -\frac{\partial U}{\partial q_i} + \sum_{j=1}^{n} g_{ij} q_j^{\cdot} - \frac{\partial f}{\partial q_i^{\cdot}} \quad (i = 1, \ldots, n;\ g_{ij} = -g_{ji}) \tag{3.7}$$

having taken $H = T + U$ as the Liapunov function, we obtain $H^{\cdot} = -2f \leqslant 0$. We assume that

$$2T = \sum_{i,j=1}^{n} a_{ij}(\mathbf{q}) q_i^{\cdot} q_j^{\cdot} \geqslant \alpha \sum_{i=1}^{n} q_i^{\cdot 2} \quad (\alpha > 0), \quad U \geqslant 0$$

According to item (4) of Theorem 5, the solution of system (3.7) is $\mathbf{q}^{\cdot}$-bounded uniformly in $\{t_0, \mathbf{q}_0, \mathbf{q}_0^{\cdot}\}$. Consequently, each solution $\{\mathbf{q}(t), \mathbf{q}^{\cdot}(t)\}$ is defined for $t \in [0, \infty)$

Differential inequalities and the comparison principle [15] may be applied to the $\mathbf{y}$-boundedness problem. Let us assume that a vector-valued function $\mathbf{V} = (V_1, \ldots, V_k)$ exists, satisfying the conditions:

1) $\mathbf{V}(t, \mathbf{x})$ and $\mathbf{V}^{\cdot}(t, \mathbf{x})$ are continuous,

2) for some l $(1 \leqslant l \leqslant k)$

$$V_1(t, \mathbf{x}) + \ldots + V_l(t, \mathbf{x}) \geqslant a(\|\mathbf{y}\|) \tag{3.8}$$

where $a(\|\mathbf{y}\|) \to +\infty$ as $\|\mathbf{y}\| \to \infty$,

3) $\mathbf{V}^{\cdot}$ by virtue of (1.1) satisfies the inequality

$$\mathbf{V}(t, \mathbf{x}) \leqslant \mathbf{f}(t, \mathbf{V}(t, \mathbf{x}))$$

while the vector-valued function $\mathbf{f}(t, \mathbf{V})$ is defined and is continuous in the region

$$t \geqslant 0, \ 0 \leqslant \|\mathbf{V}\| < +\infty$$

4) each of the functions f_s $(s = 1, \ldots, k)$ does not decrease with respect to $V_1, \ldots, V_{s-1}, V_{s+1}, \ldots, V_k$.

We denote $\boldsymbol{\alpha} = (\omega_1, \ldots, \omega_l)$ and consider the comparison system

$$\boldsymbol{\omega}^{\cdot} = \mathbf{f}(t, \boldsymbol{\omega}) \tag{3.9}$$

Theorem 6. 1) If the solutions of system (3.9) are $\boldsymbol{\alpha}$-bounded, the solutions of system (1.1) are $\mathbf{y}$-bounded uniformly in $\mathbf{x}_0$;

2) if the solutions of system (3.9) are $\boldsymbol{\alpha}$-bounded uniformly in t_0 and

$$\|\mathbf{V}(t, \mathbf{x})\| \leqslant b(\|\mathbf{x}\|)$$

the solutions of system (1.1 are $\mathbf{y}$-bounded uniformly in $\{t_0, \mathbf{x}_0\}$.

Proof. By a theorem of Wazewski [16] there exists an upper integral of system

(3.9) satisfying the inequality

$$\mathbf{V}(t,\ \mathbf{x}(t;\ t_0,\ \mathbf{x}_0)) \leqslant \boldsymbol{\omega}^+(t;\ t_0,\ \boldsymbol{\omega}_0) \tag{3.10}$$

if only $\mathbf{V}(t_0, \mathbf{x}_0) \leqslant \boldsymbol{\omega}_0$.

1) Because $\mathbf{V}$ is continuous, for each compactum K

$$\mathbf{V}(t,\ \mathbf{x}) \leqslant \boldsymbol{\varphi}_K(t) \equiv \max[\mathbf{V}(t,\ \mathbf{x}):\ \mathbf{x} \in K] \quad \text{for } t \geqslant 0,\ \mathbf{x} \in K$$

We set $\boldsymbol{\omega}_0 = \boldsymbol{\varphi}_K(t_0)$, then $\mathbf{V}(t_0,\ \mathbf{x}_0) \leqslant \boldsymbol{\omega}_0$ for $\mathbf{x}_0 \in K$. By hypothesis there exists $A(t_0,\ \boldsymbol{\omega}_0) = A_K(t_0)$ such that

$$\sum_{s=1}^{l} \omega_s^+(t;\ t_0,\ \boldsymbol{\omega}_0) \leqslant A \tag{3.11}$$

If $N(A) = N_K(t_0) > 0$ is such that $a(\|\mathbf{y}\|) > A$ follows from $\|\mathbf{y}\| > N$, then from (3.8), (3.10) and (3.11) we obtain

$$a(\|\mathbf{y}(t;\ t_0,\ \mathbf{x}_0)\|) \leqslant \sum_{s=1}^{l} V_s(t,\ \mathbf{x}(t;\ t_0,\ \mathbf{x}_0)) \leqslant \sum_{s=1}^{l} \omega_s^+(t;\ t_0,\ \boldsymbol{\omega}_0) \leqslant A$$

whence $\|\mathbf{y}(t;\ t_0,\ \mathbf{x}_0)\| \leqslant N$ for $t \geqslant t_0$.

2) We set $b_K = \sup[b(\|\mathbf{x}\|):\ \mathbf{x} \in K]$, $\omega_{s_0} = b_K\ (s = 1, \ldots, k)$. Then the numbers A_K and N_K can be chosen independent of t_0. The theorem is proved.

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BIBLIOGRAPHY

1. Rumiantsev, V.V., On the stability of motion relative to a part of the variables. Vestn. MGU, Ser. Mat. Mekh., №4, 1957.
2. Oziraner, A.S., On asymptotic stability relative to a part of the variables. Vestn. MGU, Ser. Mat. Mekh., №1, 1972.
3. Krasovskii, N.N., On the theory of A.M. Liapunov's second method for investigating the stability of motion. Dokl. Akad. Nauk SSSR, Vol.109, №3, 1956.
4. Oziraner, A.S., On stability of motion relative to a part of the variables. Vestn. MGU, Ser. Mat. Mekh., №1, 1971.
5. Gelbaum, B. and Olmsted, J., Counterexamples in Analysis. Moscow, "Mir", 1967.
6. Oziraner, A.S., Problem of the invertibility of Marchkov's theorem on asymptotic stability. Vestn. MGU, Ser. Mat. Mekh., №3, 1971.
7. Malkin, I.G., Theory of Stability of Motion. Moscow, "Nauka", 1966.
8. Rumiantsev, V.V., On the optimal stabilization of controlled systems. PMM Vol.34, №3, 1970.
9. Rumiantsev, V.V., On the stability with respect to a part of the variables, Sympos. Math. Vol.6, Meccanica non lineare e stabilita, 23 - 26 Febbrario 1970, L. – N.Y., Acad Press, 1971.
10. Halanay, A., Differential equations, stability, oscillations, time lags. N.Y., Acad. Press, 1966.
11. Yoshizawa, T., Liapunov's function and boundedness of solutions, Funkcialaj Ekvacioj, №2, 1959.
12. Yoshizawa, T., Stability theory by Liapunov's second method. J. Math. Soc. Japan, 1966.
13. Lakshmikantham, V., Vector Lyapunov's functions and conditional stability.

13. (Cont)...
J. Math. Anal. and Appl., Vol. 10, №2, 1965.
14. Demidovich, B. P., Lectures on the Mathematical Theory of Stability. Moscow, "Nauka", 1967.
15. Matrosov, V. M., On the theory of stability of motion. PMM Vol. 26, №6, 1962.
16. Wazewski, T., Systèmes des equations et des inequalités differentielles ordinaires aux deuxièmes membres monotones et leurs applications. Ann. Soc. Polon. math., Vol. 23, 1950.

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RELATIONS BETWEEN THE FIRST INTEGRALS OF A NONHOLONOMIC MECHANICAL SYSTEM AND OF THE CORRESPONDING SYSTEM FREED OF CONSTRAINTS

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We derive the necessary and sufficient conditions for obtaining the first integral of a nonholonomic system with linear homogeneous constraints from the first integral of the corresponding system freed of constraints. We present examples.

1. We consider a nonholonomic scleronomous mechanical system with the generalized coordinates $q^1, q^2, \ldots, q^n$, the doubled kinetic energy $2T = g_{\lambda\mu} q^{\cdot\lambda} q^{\cdot\mu}$ and the force function $U = U(q^\varkappa)$. The system is subject to the $n - k$ linear homogeneous constraints $\omega^p{}_\varkappa q^{\cdot\varkappa} = 0$. In what follows the Greek indices $\lambda, \mu, \nu, \ldots, \sigma$ take the values $1, 2, \ldots, n$, while the Latin ones a, b, c, d take the values $1, 2, \ldots, k$ and p, q, r take $k+1, \ldots, n$. By introducing the new variables

$$q^{\cdot\varkappa} = \alpha_a{}^\varkappa s^{\cdot a} \tag{1.1}$$

we write the equations of motion in the following form [1]:

$$Ds^{\cdot d} / dt = F^d, \qquad Ds^{\cdot d} = ds^{\cdot d} + \Gamma^d_{bc} ds^b s^{\cdot c}$$

$$F^d = G^{da} F_a = G^{da} \alpha_a{}^\varkappa Q_\varkappa = G^{da} \alpha_a{}^\varkappa \partial u / \partial q\varkappa$$

$$\Gamma^d_{cb} = G^{da} \Gamma_{a,cb}$$

$$\Gamma_{a,cb} = \Gamma_{\varkappa,\nu,\mu} \alpha_a{}^\varkappa \alpha_b{}^\mu \alpha_c{}^\nu + g_{\lambda\varkappa} \alpha_a{}^\varkappa \partial\alpha_b{}^\lambda / \partial q^\sigma \alpha_c{}^\sigma$$

The vectors α_a $(\alpha_a{}^\varkappa)$ are called the admissible vectors of the system and satisfy the condition

$$\omega_\varkappa{}^p \alpha_a{}^\varkappa = 0 \tag{1.2}$$

The matrix G^{ad} is the inverse of the matrix $G_{ab} = g_{\lambda\mu}\, \alpha_a{}^\lambda\, \alpha_b{}^\mu$. By $\Gamma_{\varkappa,\mu\nu}$ we denote the Christoffel symbols of the first kind, defined by the metric tensor $g_{\lambda\mu}$.

We consider the case when the system moves by inertia, i.e., $U = \text{const}$. As was shown in [2], in order for $\lambda_a s^{\cdot a} = c$ to be a linear integral of a nonholonomic system,